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RESONANCE EFFECTS OF ELECTROSTATIC OSCILLATIONS IN THE IONOSPHERE

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ABSTRACT

The plasma resonances observed with the Alouette topside sounder satellite occur at the plasma frequency, f_N, at multiples of the electron cyclotron frequency, f_H, and at the frequency $f_T = (f_N^2 + f_H^2)^{\frac{1}{2}}$. These resonances may be attributed to electrostatic oscillations of the ionospheric electrons in the vicinity of the satellite. The frequencies f_N and f_H correspond to oscillations along the Earth's magnetic field; the frequencies f_T and the harmonics of f_H, to oscillations across the field. The absence of responses corresponding to intermediate angles is accounted for by the greater group velocity of the pertinent electrostatic waves. These carry the energy away more rapidly and thus prevent long enduring plasma oscillations. The explanation presented, although it ignores the excitation mechanism, roughly accounts for the observed relative strengths of the resonances.

[redacted]

1. Introduction

Resonance effects in the ionosphere were first observed during rocket tests of the topside sounder technique [Knecht et al., 1961; Knecht and Russell, 1962] and more recently with the Alouette topside sounder satellite [Lockwood, 1963]. The resonances, shown in Figure 1, appear on the Alouette ionograms as persistent responses extending from the transmitter pulse at certain characteristic frequencies, often lasting for a few milliseconds. After earlier work by Knecht et al. [1961] and Lockwood [1963], the identification of the characteristic frequencies was carried out by Calvert and Goe [1963], who examined sequences of Alouette ionograms. With the variation of the electron density and the geomagnetic field along the orbit, this served to facilitate the distinction between resonances and to expose instrumental responses.

Calvert and Goe [1963], found that the resonances occurred at the following frequencies: the electron plasma frequency, f_N; multiples of the electron cyclotron frequency, f_H; the frequency $f_T = (f_N^2 + f_H^2)^{\frac{1}{2}}$, and its harmonic, $2f_T$.* Those at f_N, f_T, and low-order harmonics of f_H are always observed. The resonance at the fundamental of the cyclotron frequency is much weaker than that at the second or third harmonic, and is not consistently observed.

It was suggested by Calvert and Goe [1963] that the two major resonances at f_N and f_T result from electrostatic oscillation of the ionospheric

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* In order to avoid confusion between f_N and the f_M of Calvert and Goe [1963], the NASA Topside Sounder Working Group proposed the new notation f_T (T for transverse). We have adopted this convention.

plasma along and across the Earth's magnetic field, respectively. Lockwood [1963] interpreted the cyclotron harmonic resonances in terms of phase bunching of the electrons as they gyrate about the Earth's magnetic field lines. Johnson and Nuttal [1964] have elaborated on this and have proposed a similar excitation mechanism.

This paper extends and unifies the explanation of the resonance phenomena. Here the entire set of observed resonances is attributed to electrostatic oscillations. The nature of these oscillations, their dispersion properties, and the conditions for their persistence are considered. Finally, a comparison with the Alouette observations is made.

2. Electrostatic Oscillations

Electrostatic plasma oscillations, where the electrons of a plasma oscillate against the restoring electric force of charge separation, have been given a variety of other names. Among these are "electron-acoustic waves," "longitudinal plasma oscillations," "electron waves," "space-charge waves," or simply "plasma oscillations." In the face of such multiple terminology, we must emphasize that the term "electrostatic oscillations (or waves)" will be used for those plasma oscillations where the electric field is in the direction of the wave normal and where the wave magnetic field may be neglected. These oscillations may be treated using the electrostatic approximation to Maxwell's equations. At the frequencies where the resonance effects have been observed with Alouette, the ions remain virtually immobile and do not take part in the oscillations.

In this paper, previous work on electrostatic oscillations will be adapted for the special conditions of the upper ionosphere, and the special effects which occur there will be emphasized. In particular, the analysis of Landau [1946] and Bernstein [1958] will be used.

Electrostatic oscillations in a cold plasma were first treated by Tonks and Langmuir [1929]. They predicted standing-wave oscillations of the electrons at the plasma frequency. Later treatments, including that of Bohm and Gross [1949], incorporated the effects of random thermal motion and arrived at the conclusion that slowly propagating waves would result. Landau's [1946] treatment of these waves for a collisionless plasma, in which the ions were assumed to be stationary, was developed from the linearized Boltzmann equation and Poisson's equation. Each spatial Fourier component of the initial perturbation in the distribution function was treated separately. It was found appropriate to employ a complex frequency in the analysis, the imaginary part of which indicates attenuation of the waves generated by the initial perturbation. This attenuation phenomenon, which operates even in the absence of collisions (for a Maxwellian velocity distribution), is now known as "Landau damping."

Treatments including the effects of an ambient magnetic field have been advanced by a number of workers, including Gross [1951], Gordeyev [1952], and Bernstein [1958]. Bernstein's approach, apart from its use of the full Maxwell equations rather than the equations of electrostatics, is similar to that of Landau. In the electrostatic limit, which is used throughout this paper, and neglecting ion motions, Bernstein's results show that Landau's conclusions are considerably modified for oscillations which are not strictly along the magnetic field.

In Bernstein's analysis, an initial perturbation of the plasma can lead to two types of waves (if the motion of ions is neglected) which may usually be distinguished by their different phase velocities. The electromagnetic waves travel at velocities comparable to that of light; the electrostatic waves, at velocities comparable to the thermal velocity of electrons. In the treatment of the former, thermal motions can usually be neglected. In the treatment of the latter, it is usually permissible to use an infinite velocity of light in Maxwell's equations, or, what is equivalent, employ Poisson's equation and assume that the electric field of the wave may be derived from a potential. This procedure will be followed in the present paper to entirely exclude electromagnetic waves from consideration. The validity of the procedure will then be critically examined.

3. Dispersion in the Absence of a Magnetic Field

It was shown by Landau [1946] that each spatial Fourier component of an initial perturbation of the plasma electrons generates a longitudinal electrostatic wave of the form $\exp(st - i\underset{\sim}{k} \cdot \underset{\sim}{r})$, where $s = i\omega - \alpha$, and where ω is the angular frequency, α is a damping constant (the reciprocal of a time constant), and $\underset{\sim}{k} = 2\pi/\lambda$ is the propagation vector. For wavelengths, λ, long relative to the Debye length, h, ($hk \ll 1$), the approximate values of the components of s are:

$$\omega = \omega_N(1 + 3h^2k^2)^{\frac{1}{2}} \approx \omega_N(1 + \tfrac{3}{2}h^2k^2) \qquad (1)$$

$$\alpha = \omega_N(\pi/8)^{\frac{1}{2}}h^{-3}k^{-3} \exp(-\tfrac{1}{2}h^{-2}k^{-2}) \qquad (2)$$

where $\omega_N = (Ne^2/\epsilon_0 m)^{\frac{1}{2}}$ is the electron plasma frequency and $h = (\epsilon_0 kT/Ne^2)^{\frac{1}{2}}$ is the Debye length.

Equation 1 is the dispersion relation for electrostatic waves. It is the same as that for electromagnetic waves in the plasma, except that the velocity of light is replaced by $3^{\frac{1}{2}}h\omega_N$, which is the velocity of electrons of average thermal kinetic energy. The phase velocity of the waves is

$$v \sim \omega_N/k \tag{3}$$

and the group velocity is

$$u = \partial\omega/\partial k \sim 3h^2 k\omega_N \tag{4}$$

For long wavelengths $(hk \ll 1)$ the phase velocity is therefore higher, and the group velocity is lower, by approximately the same factor $3^{-\frac{1}{2}}(hk)^{-1}$, than the velocity $3^{\frac{1}{2}}h\omega_N$ of electrons of average thermal energy.

The time constant of Landau damping, from equation 2 is

$$T_L = (8/\pi)^{\frac{1}{2}}\, h^3k^3\omega_N^{-1}\ \exp(2^{-1}h^{-2}k^{-2}) \tag{5}$$

It should be noted that this time constant is extremely sensitive to the value of hk. For example, in ionospheric applications, where ω_N is typically about 10^7 radians/second, $hk = 0.1$ corresponds to $T_L \sim 10^{12}$ sec; $hk = 0.2$ to $T_L \sim 4 \times 10^{-4}$ sec. Landau damping is therefore insignificant for values of k less than $0.1\,h^{-1}$.

4. Relaxation of the Oscillations in the Absence of a Magnetic Field

The presence of received signals at certain frequencies for many milliseconds after the transmitter pulse must be caused by slowly decaying oscillations of the medium in the vicinity of the satellite. The long persistence of the oscillations shows that they are weakly coupled to the antenna circuit and therefore probably occur mainly in the uniform medium outside the ion sheath. Since the resonances are observed consistently, they probably are not caused by the presence of some special boundary conditions or irregularities in the medium. It seems therefore reasonable to look for the conditions under which oscillations of limited spatial extent and of long duration can occur in a uniform medium. Such conditions will be derived in this paper; they appear to be necessary for the occurrence of the resonant effects observed with Alouette.

In reality, the medium is bounded by the antenna, the body of the spacecraft, and the ion sheath around them. The problem of coupling between the antenna and the medium during transmission and reception will not be discussed, and therefore no fully quantitative predictions will be made in this paper about the absolute amplitude of the resonances.

Oscillations of limited spatial extent can be considered as a wave packet formed from a spectrum of waves with different propagation vectors $\underset{\sim}{k}$. The smallest possible size of the wave packet will be roughly determined by the width of the spectrum of spatial Fourier components. If the width of the wave number spectrum Δk is of the order of k at the center of the spectrum, then the smallest spatial extent of a wave packet formed from such waves is about $2\pi/k$. Since the direction of the group velocity is that of $\underset{\sim}{k}$, such a wave packet will not propagate in a single direction but rather will spread in all directions. It is reasonable to assume that the spreading will be controlled by the magnitude of the group velocity. Substantial spread will occur for times greater than the time T_G obtained by dividing the size of the wave packet $2\pi/k$ by the group velocity u. For electrostatic waves in the absence of a magnetic field

$$T_G = 2\pi k^{-1} u^{-1} = \tfrac{2}{3}\pi h^{-2} k^{-2} \omega_N^{-1} \tag{6}$$

where the value of u from equation 4 has been substituted. It is convenient to express T_G in terms of the number τ of wave periods $2\pi/\omega$. Substantial spreading will therefore occur after

$$\tau = T_G \omega / 2\pi = v/u \tag{7}$$

periods where $v = \omega/k$ is the phase velocity. This parameter will be used for the rest of the paper as a measure of the time scale of the oscillations. For electrostatic waves in the absence of a magnetic field the combination of equations 6 and 7 yields

$$\tau \sim (3h^2 k^2)^{-1} \tag{8}$$

Equation 7 indicates that oscillations of limited spatial extent and of long duration occur if the phase velocity is much larger than the group velocity. In the special case of electrostatic waves in the absence of a magnetic field, this condition is satisfied according to equation 8 if hk is much smaller than unity. It may be easily ascertained that for $hk < 0.1$ the time T_G given by (6) is much smaller than the time T_L given by (5). The persistence of the oscillations is thus determined almost entirely by dispersion and the decay caused by Landau damping is insignificant.

5. Dispersion in the Presence of a Magnetic Field

The starting point here is Bernstein's [1958] equation 40, which relates the complex angular frequency, -is, to the propagation vector, $\underset{\sim}{k}$. It was derived from the collisionless, linearized Boltzmann equation and Maxwell's equations. It involves the approximation that the velocity of light and the mass of the ions are infinitely large. It thus pertains only to electrostatic oscillations of the electrons. This (complex) dispersion equation, which corresponds to equations 1 and 2 of the previous section, is

$$1 + h^2k^2 = iq\omega_H \exp(-k^2\rho^2\sin^2\theta) \sum_{m=-\infty}^{\infty} I_m(k^2\rho^2\sin^2\theta) \cdot \int_0^{\infty} \exp\left[-\tfrac{1}{2}\omega_H{}^2k^2\rho^2\cos^2\theta t^2 - i\omega_H(q+m)\, t\right] dt \ , \qquad (9)$$

where I_m is the modified Bessel function, θ is the angle between k and the magnetic field, m is an integer, ω_H is the electron cyclotron frequency, $\omega + i\alpha = -is$ is the complex angular frequency, $q = -ks/\omega_H$, $q_N = \omega_N/\omega_H$, and $\rho = q_N h$ is the cyclotron radius of electrons of average kinetic energy.

The integrals in equation 9 may be transformed by the identity

$$i\int_0^\infty \exp(-\tfrac{1}{4}b^2t^2 - iat)\,dt = b^{-1}\exp(-a^2/b^2)(i\pi^{\frac{1}{2}} + 2\int_0^{a/b} e^{t^2}\,dt) \qquad (10)$$

where $a = \omega_H(q + m)$ and $b = 2^{\frac{1}{2}}k\rho\omega_H\cos\theta$. Notice that the rest of equation 9 is real except for the factor q which multiplies the sum. Thus the phase angle of q will be the complement of that of the sum, and is controlled by the integrals (10). The condition for a small phase angle, which corresponds to weak Landau damping, is that the integral on the right of equation 10 dominate the $i\pi^{\frac{1}{2}}$ imaginary term. In other words, a/b must be much larger than about 0.7 in absolute value. This means that the condition for weak Landau damping is

$$|q + m| \ll |k\rho\cos\theta| \;, \qquad (11)$$

for all m. Waves perpendicular to the magnetic field ($\theta = \pi/2$) will be undamped. Those at other angles will suffer only weak Landau damping if the wavelength is sufficiently long ($k\rho \ll 1$), provided that the frequency is not close to a multiple of the cyclotron frequency ($q + m \neq 0$). In the case of interest here, where condition (11) is met, we may discard the imaginary term on the right of equation 10 and confine ourselves to real q. Equation 9 then takes the form

$$1 + (\omega_H^2/\omega_N^2)\,k^2\rho^2 = 2^{\frac{1}{2}}q(k\rho\cos\theta)^{-1}\exp(-k^2\rho^2\sin^2\theta)\sum_{m=-\infty}^{\infty} I_m(k^2\rho^2\sin^2\theta)$$
$$\exp\left[-(q + m)^2/2k^2\rho^2\cos^2\theta\right]\int_0^{2^{-\frac{1}{2}}(q+m)\,k^{-1}\rho^{-1}\sec\theta} e^{t^2}\,dt \qquad (12)$$

In order to study the solutions of (12) for small values of $k\rho$ it is convenient to expand this dispersion equation in powers of $k\rho$, using the following formulas:

$$2e^{-x^2} \int_0^x e^{t^2}\, dt = \pi^{-\frac{1}{2}} \sum_{\nu = 0} x^{-(2\nu + 1)} (\nu - \tfrac{1}{2})! \tag{13}$$

$$I_m(y) = \sum_{j = 0}^{\infty} (y/2)^{m + 2j} [j!(m + j)!]^{-1} \tag{14}$$

The former is an asymptotic expansion [Fried and Conte, 1961] and the latter is the Taylor series for the modified Bessel function [Watson, 1948]. If only first and second order terms in $k^2\rho^2$ are retained in the expansion of (12) then the approximate dispersion equation is

$$\frac{\omega_H^2}{\omega_N^2} = q_N^{-2} = \frac{\sin^2\theta}{q^2 - 1} + \frac{\cos^2\theta}{q^2} + k^2\rho^2\left[\frac{3\sin^4\theta}{(q^2 - 1)(q^2 - 4)} + \frac{\sin^2\theta\cos^2\theta(6q^4 - 3q^2 + 1)}{q^2(q^2 - 1)^3}\right] + k^2\rho^2\left[\frac{3\cos^4\theta}{q^4}\right] \tag{15}$$

If the $k^2\rho^2$ term is neglected altogether then a quadratic equation in q is obtained whose solution is

$$q^2 = \tfrac{1}{2}\left\{1 + q_N^{2} + [(1 - q_N^2)^2 + 4q_N^2 \sin^2\theta]^{\frac{1}{2}}\right\} \tag{16}$$

This condition is identical to the well-known condition $X = (1 - Y^2)/(1 - Y_L^2)$ [Ratcliffe, 1959] for the infinity of the refractive index of electromagnetic waves in a plasma. The frequencies given by (16) are usually called resonant frequencies [Spitzer, 1962]. They are shown by Figure 2 as functions of θ for different values of q_N. It is to be noted that only the values of q for $\theta = 0$ and $\theta = \pi/2$ are actually observed in Alouette records; these are $q = 1$, $q = q_N$ (or $f = f_H$, $f = f_N$) for $\theta = 0$; and $q = (1 + q_N^2)^{\frac{1}{2}}$ (or $f = f_T$) for $\theta = \pi/2$. The theoretical explanation of the absence of resonant frequencies which could be attributed to other angles is discussed in the following section.

In the special case of $\theta = \pi/2$, equation 16 does not represent the only solutions for small values of $k\rho$. Although equation 15 does not indicate all the additional solutions for $\theta = \pi/2$, its solution

$$q^2 = \frac{1}{2}\left\{5 + q_N^2 \pm [(q_N^2 - 3)^2 + 12k^2\rho^2 q_N^2]^{\frac{1}{2}}\right\} \tag{17}$$

results in two values for q. For small values of $k\rho$, one of these approaches $(q_N^2 + 1)^{\frac{1}{2}}$, which is one of the solutions noted above. The other solution approaches 2 and thus represents a frequency close to the second harmonic of the electron cyclotron frequency. Examination of the full equation 12 with $\theta = \pi/2$ shows that similar solutions also exist near all the other harmonics of the cyclotron frequency. These will be discussed in more detail in the following section.

The results of the present section are that the solutions of the dispersion equation for small values of $k\rho$, for $\theta = 0$ and $\theta = \pi/2$, coincide approximately with the frequencies of the resonances observed with the Alouette satellite. The decay time of these resonances and the reason for the absence of observations corresponding to solutions of the dispersion equation for intermediate angles are discussed in the following section.

6. Relaxation of the Oscillations in the Presence of a Magnetic Field

Conditions for the occurrence of oscillations of limited spatial extent and of long duration in a uniform medium are modified considerably by the presence of a magnetic field. The arguments about the unimportance of Landau damping are not changed essentially by the presence of a magnetic field although they are strengthened by the complete absence of Landau damping for $\theta = 90^\circ$ (propagation in a direction normal to the magnetic field). The arguments about the spreading of a wave packet are, however, changed considerably by the anisotropic nature of the medium. In the absence of a magnetic field a wave packet was found to decay with almost monochromatic oscillations in the vicinity of the plasma frequency (subject to the condition $kh \ll 1$). In the presence of a magnetic field Figure 2 shows that, for small values of k, any single direction of the vector $\underset{\sim}{k}$ corresponds to two frequencies. The harmonics of the cyclotron frequency must be added to these frequencies in the special case when the vector $\underset{\sim}{k}$ is normal to the magnetic field.

The spreading of the wave packet formed by such Fourier components is again determined by the group velocity which, in an anisotropic medium, is given by $\partial\omega/\partial\underset{\sim}{k}$, that is, the gradient of the scalar quantity ω in $\underset{\sim}{k}$-space. Figure 3a shows curves of constant $q = \omega/\omega_H$ in this space. The surfaces of constant q are obtained from these curves by rotation about the horizontal axis (parallel to the magnetic field). The group velocity is perpendicular to these surfaces and is indirectly proportional to the perpendicular distance between them. For $kh \ll 1$ the group velocity is thus almost perpendicular to $\underset{\sim}{k}$ for all values of the angle θ that are not in the vicinity of $\theta = 0$ or $\theta = \pi/2$.

Figure 3a represents the electrostatic approximation of the dispersion equation 15. It shows that the curves of constant q become almost radial near the origin so that a given frequency corresponds to a certain angle between the wave normal and the magnetic field (c.f. equation 16). In reality the electrostatic approximation breaks down close to the origin. The shape of the curves in this vicinity (i.e. well inside the small circle of Figure 3a) is shown by Figure 3b which represents the cold plasma approximation and is based on the Appleton-Hartree equations [Ratcliffe, 1959]. The significance of the curves of Figure 3b will be discussed later in the paper; at present only the electrostatic approximation will be considered.

In order to discuss the relaxation of oscillations corresponding to angles between $\theta = 0$ and $\theta = \pi/2$, it is necessary to carefully re-evaluate the quantity T_G, that is, the scale of the wave packet divided by the group velocity. Since the constant-frequency curves are almost radial for this case, the group velocity is approximately $u = k^{-1}\partial\omega/\partial\theta$ and it is directed almost perpendicular to k. Therefore, the pertinent scale of the wave packet is that also perpendicular to $\underset{\sim}{k}$. The width of the spatial spectrum corresponding to a given angular width $\Delta\theta$ is $\Delta k = k\Delta\theta$ (the approximate range of the azimuthal component of $\underset{\sim}{k}$). Thus the scale of the wave packet is $2\pi/k\Delta\theta$. The angular width is determined by the bandwidth $\Delta\omega$ of the sounding system, given by $\Delta\omega = (\partial\omega/\partial\theta)\Delta\theta$. Therefore,

$$T_G = \frac{2\pi/k\Delta\theta}{k^{-1}\partial\omega/\partial\theta} = 2\pi/\Delta\omega$$

In the case of Alouette this indicates that persistent oscillations should be observed no longer than the relaxation time of the receiver, also equal to $2\pi/\Delta\omega$.

The situation is different when the pass-band of the satellite includes a frequency corresponding to $\theta = 0$ or $\theta = \pi/2$. In these cases the group velocity is considerably smaller than in the intermediate case, and it is directed parallel to $\underset{\sim}{k}$. Once again the appropriate scale is given by $2\pi/k$ for Δk the same order as k. Although this scale is smaller than in the intermediate case, the greatly reduced value of the group velocity (for sufficiently small k), gives T_G a considerably larger value and thus accounts for the persistence of oscillations at the limiting angles. As in the absence of the magnetic field, this persistence can again by measured by τ, the ratio of phase velocity to group velocity. This ratio becomes infinitely large in the electrostatic approximation as k approaches zero.

Details of the derivations of approximate expressions for the relaxation time τ measured in cycles and for the angular range θ_m of the Fourier components that form the slowly spreading wave packet, are given in the Appendix for the resonances near $\omega = \omega_N$, $\omega = \omega_H$, $\omega = \omega_T = (\omega_N^2 + \omega_H^2)^{\frac{1}{2}}$, and $\omega = n\omega_H$ where n is an integer and $n \geq 2$. The resulting expressions are given in Table I. Also given in Table I are the calculated fractional deviations d_f of the oscillations from the nominal frequency (without the sign). The table shows that this deviation is only significant for the ω_H resonance, for which $d_f \sim 0.08$ for $\tau = 1000$. For the other resonances

$d_f < 0.01$ for $\tau > 100$ and therefore the frequency deviation is not likely to exceed the observational error. Details of the calculation of d_f are not given since they form a straightforward extension of the analysis given in the Appendix.

7. The Validity of the Electrostatic Approximation

In the previous sections the equations of electrostatics were used instead of the complete Maxwell equations. The errors caused by that approximation can be estimated by using the quasi-hydrodynamic plasma equations (instead of the collisionless Boltzmann equations) combined with the full Maxwell equations. Such a procedure, although it ignores the resonances at the harmonics of the cyclotron frequency, should be quite realistic for the other resonant frequencies. In a direction parallel to the magnetic field the introduction of the full Maxwell equation causes no change in the results; the "electrostatic" waves and the "electromagnetic" waves propagate quite independently in that direction. In all other directions the "electrostatic" or "plasma" wave is a continuation of one of the "electromagnetic" magnetoionic modes. This is illustrated by Figure 3b which is based on the Appleton-Hartree equations and represents the form of the curves of Figure 3a very close to the origin. Figure 3b shows that the curves do not continue to approach the origin radially but turn to intersect the axis perpendicular to the magnetic field (the $\theta = \pi/2$ axis).

The dispersion equation for $\theta = \pi/2$ is given explicitly by Ginzburg [1961, equation 12.8] and can be used to derive $\tau = v/u$, the duration of resonant oscillations, measured in cycles. It is then found that τ does not increase indefinitely with decreasing k, as predicted by the electrostatic approximation but reaches a maximum value given by

$$\tau_{max} = \frac{\omega_H^2 + \omega_N^2}{3^{\frac{1}{2}}\omega_H\omega_N} \frac{c}{h\omega_N} \tag{18}$$

If, therefore, ω_H and ω_N are of the same order of magnitude then the largest value of τ is about equal to the ratio of the velocity of light c to the mean thermal velocity ($\sim h\omega_N$) of the electrons. This means that at the hybrid frequency, f_T, a marked resonance is then only possible if the ratio $c/h\omega_N$ is very large. In the plasma encountered by Alouette this ratio is probably always greater than 1000 and thus the hybrid resonance is not inhibited and the electrostatic approximation is probably satisfactory.

8. Physical Nature of the Oscillations and their Excitation

The theory of Section 6 is linear in its nature. It may seem paradoxical that a linear theory should predict oscillations at harmonics of the cyclotron frequency since harmonics are usually associated with non-linear processes. It must be realized that, although the oscillations were considered to be infinitesimally small, the thermal motion causes the electrons to gyrate with finite cyclotron radii. Indeed, cold plasma theory does not predict any special effects at the harmonics of the cyclotron frequency. These effects depend on thermal motion. A bunching of the gyrating electrons is caused by the alternating electric field (normal to the magnetic field) which in turn is caused by the bunched gyrations. The present theory shows that self-consistent oscillations of this type occur very near to the harmonics of the cyclotron frequency but that the frequency of the fundamental mode is approximately f_T and not f_H. It is close to the cyclotron frequency only for a very tenuous plasma.

The present explanation of the oscillations near the harmonics of the electron cyclotron frequency is not complete in the sense that it only explains the persistence of the oscillations but not the mechanism of their generation. It differs in this respect from the work of Lockwood [1963] and Johnston and Nuttal [1964] who tried to explain the generation of oscillations near the cyclotron harmonics without explaining their persistence. Their suggested explanations of the generation process are, however, only semi-quantitative and further work is required on the generation process.

Further light is thrown on the nature of the excitation process by the observation of a resonance at the frequency $2f_T$. It appears very likely that the actual oscillations in this case occur at the frequency f_T, and that the second harmonic $2f_T$ is produced in the receiver. Oscillations of the medium at the frequency f_T can therefore apparently be excited by the transmitter even when it is tuned to a frequency $2f_T$ or presumably to any other arbitrary frequency. It thus would seem that oscillations of considerable amplitude are excited impulsively at frequencies different from the frequency of the transmitter. The actual excitation process is probably strongly non-linear and is not discussed here.

Other workers [Warren and Nelms, 1964] described the Alouette resonances in terms of the cold plasma approximation (the Appleton-Hartree equations). In a sense such a description is not always essentially different from the one given here since the electromagnetic waves in a cold plasma become almost longitudinal in the presence of an external magnetic field and a large refractive index. The waves are then really electrostatic in nature and it is essential to take the effects of thermal motion into account to provide their full description. If the thermal motions are not taken into account, then no lower limit is obtained for the wavelength of lightly damped and long persistent oscillations. It was shown, however, in the present paper that the oscillations can only have long decay time if their wavelength greatly exceeds the Debye length; the exact conditions are given in Table I. Conditions of this type cannot be obtained from the Appleton-Hartree equations; theories based on them such as that of Warren and Nelms [1964] and the calculations of antenna impedance in the presence of a magnetic field by Kaiser [1962] cannot therefore be used for the explanation of the Alouette resonances without bearing this limitation in mind.

Nuttal [1964] attempts to treat the excitation of the resonances at ω_N and ω_T in a quantitative manner. He uses first the cold plasma approximation and then the collisionless Boltzmann equation; he neglects sheath effects and thus is able to use linear theory. He does not, however, obtain detailed quantitative results that could be compared with the observations. Further theoretical work, preferably with still more realistic assumptions, is clearly required.

9. Comparison with Observations

The most obvious agreement between the theory and the Alouette observations is the agreement of the predicted with the observed frequencies. Furthermore, as will be shown below, there is also general agreement between the observed strengths of the resonances and crude predictions based on the present theory.

Resonances are observed for each of the frequencies predicted by the theory. Furthermore, it is felt that all other observed responses can be attributed to auxiliary mechanisms. These additional responses are found to occur at the following frequencies: (1) the harmonic of f_T, and much less often, the harmonic of f_N; (2) 1 Mc/s below predicted resonances; and (3) at random frequencies above the F layer penetration frequency. As suggested earlier, the responses at $2f_T$ and $2f_N$ (item 1) might be attributed to impulsive excitation of oscillations at the fundamental frequency by the 100 microsecond transmitter pulse, and then reception by harmonic generation in the early broadband stages of the receiver. This, however, leaves unexplained the observation that $2f_T$ is much more common than $2f_N$. The image responses (item 2) have been adequately explained by Southern [private communication, 1964] as harmonic transmission followed by image reception in the second-IF circuits of the receiver. Finally, the responses at random frequencies (item 3), which have been found to occur most often over populated areas, are attributed to interference by ground-based transmitting stations. To our knowledge no other prominent resonances appear on the Alouette ionograms although other points of view have been expressed [Fitzenreiter and Blumle, 1964].

Agreement between the observed and the predicted strength of the resonances is more difficult to establish. First of all, theoretical predictions can, at their best, only be rough estimates in the absence of a full theory of excitation and of antenna behavior. Secondly, the response of the receiver to input signals is complicated, even at a single frequency, by the automatic volume control circuit. In addition the response is frequency dependent and falls off rapidly below about 1 Mc/s.

The duration of the response on the Alouette ionogram provides the most easily-obtained rough estimate for the observed strength of the resonances. This duration must depend (ignoring the excitation process) not only on the predicted relaxation time $2\pi\tau/\omega$ but also on the angular range θ_m of the waves which form the slowly spreading wave packet. The quantities τ and θ_m are given by Table I for the different resonances. While neither of these two quantities nor any combination of them can be regarded as the predicted strength of a resonance--especially as the value of k is unknown--certain conclusions can nevertheless be reached. First of all, it appears likely that the smallest value of k that is appreciably excited is somehow related to the largest linear dimension of the antenna. With antennas as large as those on Alouette, the largest appreciably-excited spatial Fourier component corresponds to a value of kh probably smaller than 10^{-3}. This hardly sets a practical limit on the value of τ. Since Table I shows that θ_m decreases with increasing τ, it seems likely that the practical limit on τ is set by the minimum value of θ_m that will produce measurable oscillations for a given mode. For this reason τ is also given in Table I in terms of θ_m and q_N. The table shows that the expression for τ is a product of separate functions of θ_m and q_N. With the *ad hoc* assumption that θ_m is constant for each of the resonances, the function of q_N could be interpreted as the relative predicted duration in cycles for each mode separately as q_N varies with latitude. However, since the values of θ_m are unknown, it is not possible to estimate the ratios of strength between the different modes.

A special study of the Alouette resonances was carried out for comparison with the functions of column three in the table. In this study, the duration of the resonances (the lengths on the record) were recorded and averaged for each value of $q_N = \omega_N/\omega_H$. These averages, with each point representing one to three dozen measurements, are presented in Figure 4. In order to indicate the spread in the data, flags were added which bracket the central 50% of the observations (the 25 and 75 percentile limits). The dashed curves in these figures give the observational limits imposed by the weakest observable resonance (bottom), off-scale resonances (top), and the 1 Mc/s low-frequency limit of Alouette (left or right). Superimposed on the data points are the variations predicted in the table. These curves have been registered vertically to obtain the best agreement.

Figures 4a and 4b for f_N and f_H represent reasonable agreement with the theory and support its most positive prediction. This is the inhibition of each of these two resonances at $q_N = 1$. It is, however, unfortunate that the 1 Mc/s lower threshold of Alouette (imposed by the Antenna-matching network) coincides with this condition and thus obscures in each case the recovery on the other side. Figures 4c and 4d for f_T and $2f_H$ show good agreement for values of q_N above 2. The source of the discrepancies for lower values (at higher latitudes) is not known. Figure 4e for $3f_H$ shows surprisingly good agreement throughout the range except for the single point at $q_N = 3.05$. Finally, Figure 4f for $4f_H$ represents agreement with the theory. In all, the agreement with observations in Figure 4 is reasonably good in light of the ignored excitation mechanism and receiver response.

10. Conclusions

The resonances observed by the topside sounders may be attributed to electrostatic oscillation of the ionospheric plasma. This represents an extension of the interpretation of Calvert and Goe [1963] to include the cyclotron resonances.

The resonant frequencies f_N and f_H correspond to oscillations approximately along the ambient magnetic field; f_T, $2f_H$, $3f_H$, etc., to oscillations across the field. The oscillations at intermediate angles (with frequencies between f_N and f_T) do not persist for long in the vicinity of the sounder because their energy spreads more rapidly on account of their greater group velocity.

Collisional damping is negligible in the topside ionosphere, as is Landau damping for $hk < 0.1$. The principal mechanism controlling the relaxation of the oscillations appears to be the spreading of energy by propagation away from the sounder. A consideration of this mechanism in Section 6 led to the relaxation time, the angular tolerances, and the frequency tolerances shown in Table I for the various resonances. The expressions of Table I were used to estimate the resonance strengths and these estimates agreed resonably well with the Alouette observations where comparison was possible. Both the theory and the observation will have to be improved before a more detailed comparison can be made.

Acknowledgements

We acknowledge helpful discussions with Drs. T. E. VanZandt, D. Walsh, F. S. Johnson, J. Nuttal, and W. Deering. Thanks are due to Miss J. Ligon for assistance with the computations. The research was supported by the National Aeronautics and Space Administration under NSG-269-62 (J. A. Fejer) and S-2481-G (W. Calvert).

APPENDIX

Calculations of τ and θ_m for the Different Resonances

(a) Oscillations near the plasma frequency f_N: $\theta \sim 0$, $q \sim q_N$

For small values of h^2k^2 and for $\theta = 0$, equation 15 may be shown to be equivalent to equation 1 and therefore the ratio of the phase velocity to the group velocity is given by

$$\tau = v/u = (3h^2k^2)^{-1} \tag{19}$$

just as if the magnetic field were absent. An electrostatic oscillation at a frequency very close to the plasma frequency and with duration of about $(3h^2k^2)^{-1}$ periods is therefore to be expected. Whereas in the absence of a magnetic field all Fourier components with wave numbers of order k were responsible for the oscillation, the magnetic field restricts the angular range of the vectors to a narrow cone whose axis lies along the magnetic field. The maximum angle θ_m between $\underset{\sim}{k}$ and the magnetic field may be roughly estimated as the angle where the group velocity obtained from (16) with the aid of the formula (derived from $\underset{\sim}{u} = \partial\omega/\partial\underset{\sim}{k}$)

$$u = k^{-1}\partial\omega/\partial\theta \tag{20}$$

is approximately equal to $3h^2k\omega_N$, the value of the group velocity in the absence of a magnetic field given by equation 4. The value obtained in this manner is

$$\theta_m = \frac{\tau^{-1}|\omega_N^2 - \omega_H^2|}{\omega_H^2} \tag{21}$$

This equation shows that the resonant oscillations are restricted to a relatively narrow angle about the magnetic field and that the angle decreases as the wavelength, and therefore the duration of the oscillations, increases. The equation also shows that no resonant oscillations can occur when the plasma frequency is equal to the electron cyclotron frequency.

(b) Oscillations in the vicinity of the cyclotron frequency f_H: $\theta \sim 0$, $q \sim 1$

It will be recalled that equation 15, which predicts the resonance for $q \sim 1$ at $\theta = 0$, is invalid for integral values of q because the condition (11) is not satisfied. If $|q - 1|/|k\rho\cos\theta|$ exceeds about 8 (this value is not critical) then Landau damping becomes negligible and equation 15 becomes applicable. If the approximate value of q from equation 16 is substituted into the limiting condition $|q - 1|/|k\rho\cos\theta| = 8$, and the resulting equation is solved for θ, then the expression

$$\theta_e = 4\omega_N^{-1}(k\rho|\omega_N^2 - \omega_H^2|)^{\frac{1}{2}} \tag{22}$$

is obtained for the angle θ_e, between $\underset{\sim}{k}$ and the magnetic field, which has to be exceeded for negligible Landau damping. Since the angular frequency ω, given in terms of equation 16 is approximately

$$\omega \sim \omega_H + \tfrac{1}{2}\omega_H\omega_N^2\theta^2/|\omega_H^2 - \omega_N^2| \,, \tag{23}$$

the derivative $d\omega/d\theta$ will not be constant over the receiver bandwidth which corresponds to an angle much larger than θ_e. The arguments used for an arbitrary angle θ at the beginning of this section cannot therefore be applied for the total receiver bandwidth, and the relaxation time of the oscillations will be determined by that group of waves between, say, θ_e and $\frac{3}{2}\ \theta_e$ for which $\partial\omega/\partial\theta = \theta\omega_H\omega_N^2/|\omega_H^2 - \omega_N^2|$ varies relatively little. The width of the angular frequency band occupied by these waves is approximately $\frac{5}{8}\omega_H\omega_N^2\theta_e^2/|\omega_H^2 - \omega_N^2|$ and the duration of the oscillations will be approximately the reciprocal of the frequency bandwidth. If θ_e from (22) is substituted then

$$\tau \sim 10^{-1}k^{-2}\rho^{-2} \sim 10^{-1}k^{-2}h^{-2}\omega_H^2/\omega_N^2 \tag{24}$$

is obtained for the duration of the oscillations, expressed in cycles. If ω_H and ω_N do not differ greatly, then equations 19 and 24 yield durations of the same order of magnitude. If the angular range of the waves responsible for the resonance is taken as $\theta_m = \frac{1}{2}\theta_e$ and is expressed in terms of τ with the aid of equations 22 and 24 then the expression

$$\theta_m = \left(\tfrac{5}{8}\ \tau\right)^{-\frac{1}{4}} \omega_N^{-1}\left|\omega_N^2 - \omega_H^2\right|^{\frac{1}{2}} \tag{25}$$

is obtained.

It should be noted that the angular range given by (25), like the angular range given by (21), contracts to zero when the plasma frequency is equal to the electron cyclotron frequency.

(c) Oscillations near the hybrid frequency f_T: $\theta \sim \pi/2$, $q \sim (1 + q_N^2)^{\frac{1}{2}}$

The group velocity for these oscillations may be found from equation 15 after the substitution $\theta = \pi/2$. The result for the relaxation time, measured in periods, is

$$\tau = v/u = (3h^2k^2)^{-1}(\omega_N^2 + \omega_H^2)|\omega_N^2 - 3\omega_H^2|/\omega_N^4 \tag{26}$$

This equation becomes identical to (19) for $\omega_H = 0$. It is seen that the relaxation time of the oscillations becomes very short if $\omega_N^2 \sim 3\omega_H^2$ or $\omega_T = (\omega_N^2 + \omega_H^2)^{\frac{1}{2}} \sim 2\omega_H$; the same situation would be found for $\omega_T = 3\omega_H$, $4\omega_H$ ... if the more accurate equation 12 were used instead of (15).

The approximate angular range θ_m of the waves may be found by equating the group velocity given by equation 26 with that obtained from the combination of equations 16 and 20. The resulting equation is

$$\theta_m = \tau^{-1}(\omega_N^2 + \omega_H^2)^2/\omega_N^2\omega_H^2 \tag{27}$$

(d) Oscillations at harmonics of the electron cyclotron frequency nf_H: $\theta \sim \pi/2$, $q \sim n$ where $n \geq 2$ is an integer

Equation 15 is not directly applicable in this case but its terms not involving k can be retained if $k^{-2}\rho^{-2}$ times the $m = -n$ term of the sum in equation 12 is added on the right hand side of (15). For $\theta \sim \pi/2$ the resulting equation is

$$\frac{\omega_H^2}{\omega_N^2} - \frac{1}{n^2 - 1} = n(k\rho)^{2n - 3}\, 2^{-n + \frac{1}{2}}\, (n!)^{-1} \sec\theta \exp\left[-\left(\frac{q - n}{2^{\frac{1}{2}}k\rho\cos\theta}\right)^2\right] \int_0^{\frac{q - n}{2^{\frac{1}{2}}k\rho\cos\theta}} e^{t^2}\, dt \tag{28}$$

In the special case of $\theta = \pi/2$ the first term of the asympototic expansion (13) may be used to obtain from equation 28 the relation

$$\frac{\omega_H^2}{\omega_N^2} - \frac{1}{n^2 - 1} = \frac{(k^2\rho^2)^{n - 1}}{2^n(n - 1)!}\, \frac{1}{q - n} \tag{29}$$

which has the solution

$$q = (n + 2)^{-n}\, \frac{(k^2\rho^2)^{n - 1}}{(n - 1)!} \left[\frac{\omega_H^2}{\omega_N^2} - \frac{1}{n^2 - 1}\right]^{-1} \tag{30}$$

If $k^2\rho^2 \ll 1$ and the equation $\omega_H^2/\omega_N^2 = (n^2 - 1)^{-1}$, or $(\omega_H^2 + \omega_N^2)^{\frac{1}{2}} = n\omega_H$, is not nearly satisfied then q will be very nearly equal to n, as was assumed. Differentiation of (30) with respect to k then results in the equation

$$\tau = v/u = 2^{n - 1}\, n(n - 2)!(k^2\rho^2)^{-(n - 1)} \left|\frac{\omega_H^2}{\omega_N^2} - \frac{1}{n^2 - 1}\right| \tag{31}$$

The angular range θ_m may be determined from the limiting condition (analogous to a similar condition used in Section b) $|q - n|/|k\rho\cos\theta| = 8$; Landau damping will then be sufficiently small for angles smaller than θ_m since condition (11) is satisfied for these angles. After the elimination of $k\rho$ with the aid of equation 31 the limiting expression takes the form

$$\theta_m = 2^{-9/2} n(n-1)^{-1} \tau^{-\frac{2n-3}{2n-2}} \left[n(n-2)! \left| \frac{\omega_H^2}{\omega_N^2} - \frac{1}{n^2-1} \right| \right]^{-\frac{1}{2n-2}} \tag{32}$$

REFERENCES

Bernstein, I. B., Waves in a plasma in a magnetic field, Phys. Rev., 109 (1), 10-21, 1958.

Bohm, D., and E. P. Gross, Theory of plasma oscillations, Phys. Rev., 75, 1851-1876, 1949.

Calvert, W., and G. B. Goe, Plasma resonances in the upper ionosphere, J. Geophys. Res., 68 (22), 6113-6120, 1963.

Fitzenreiter, R. J., and L. J. Blumle, Analysis of topside sounder records, J. Geophys. Res., 69 (3), 407-415, 1964.

Ginzburg, V. L., Propagation of Electromagnetic Waves in Plasma, Gordon and Breach, New York, 1961, pp. 16-17.

Gordeyev, G. V., Plasma oscillations in a magnetic field, Zhur, Eksp. i. Teorte, Fiz., 23, 660, 1952.

Gross, E. P., Plasma oscillations in a static magnetic field, Phys. Rev., 82 (2), 232-242, 1951.

Johnston, T. W., and J. Nuttal, Cyclotron harmonic signals received by the Alouette topside sounder, J. Geophys. Res., 69 (11), 2305-2314, 1964.

Kaiser, T. R., The admittance of an electric dipole in a magnetoionic environment, Planetary Space Sci., 9, 639-657, 1962.

Knecht, R. W., and S. Russell, Pulsed radio soundings of the topside of the ionosphere in the presence of spread F, J. Geophys. Res., 67 (3), 1178-1182, 1962.

Knecht, R. W., T. E. VanZandt, and S. Russell, First pulsed radio soundings of the topside of the ionosphere, J. Geophys. Res., 66 (9) 3078-3081, 1961.

Landau, L., On the vibrations of the electronic plasma, J. Phys. USSR, 10 (1), 25-34, 1946.

Lockwood, G. E. K., Plasma and cyclotron spike phenomena observed in topside ionograms, Can. J. Phys., 41, 190-194, 1963.

Nuttal, J., Theory of collective spikes observed by the Alouette topside sounder, RCA Victor Research Report 7-801-29c, Montreal, Canada, April 1964.

Ratcliffe, J. A., The Magnetoionic Theory and Its Application to the Ionosphere, Cambridge University Press, London, 1959.

Tonks, L., and I. Langmuir, Oscillations in ionized gases, Phys. Rev., 33, 195-210, 1929.

Warren, E. S., and G. L. Nelms, Dispersion of pulses of radio waves originating in the ionosphere, Presented to American Geophysical Union, April 1964.

CAPTIONS

1. Alouette ionogram showing the various resonances. Recorded at 1437 UT, April 7, 1963.

2. The two resonance solutions $q = \omega/\omega_H$ of equation 16 as functions of the angle θ between the wave normal and the magnetic field, for two different values of the parameter $q_N = \omega_N/\omega_H$.

3. Curves illustrating the surfaces of constant $q = \omega/\omega_H$ in $\underset{\sim}{k}$-space for $q_N = 1.25$ ($q_T = 1.60$). These curves are based on (a) the collisionless Boltzmann equation and the electrostatic approximation; and (b) the Appleton-Hartree equation. The latter illustrates the behavior near the origin.

4a. Alouette observations of the duration of the resonances at the resonant frequency f_N. The solid curves give, as functions of q_N, the variations predicted in Table I, $\tau(\theta_M = \text{const}, q_M)$, times constant factors adjusted for best agreement.

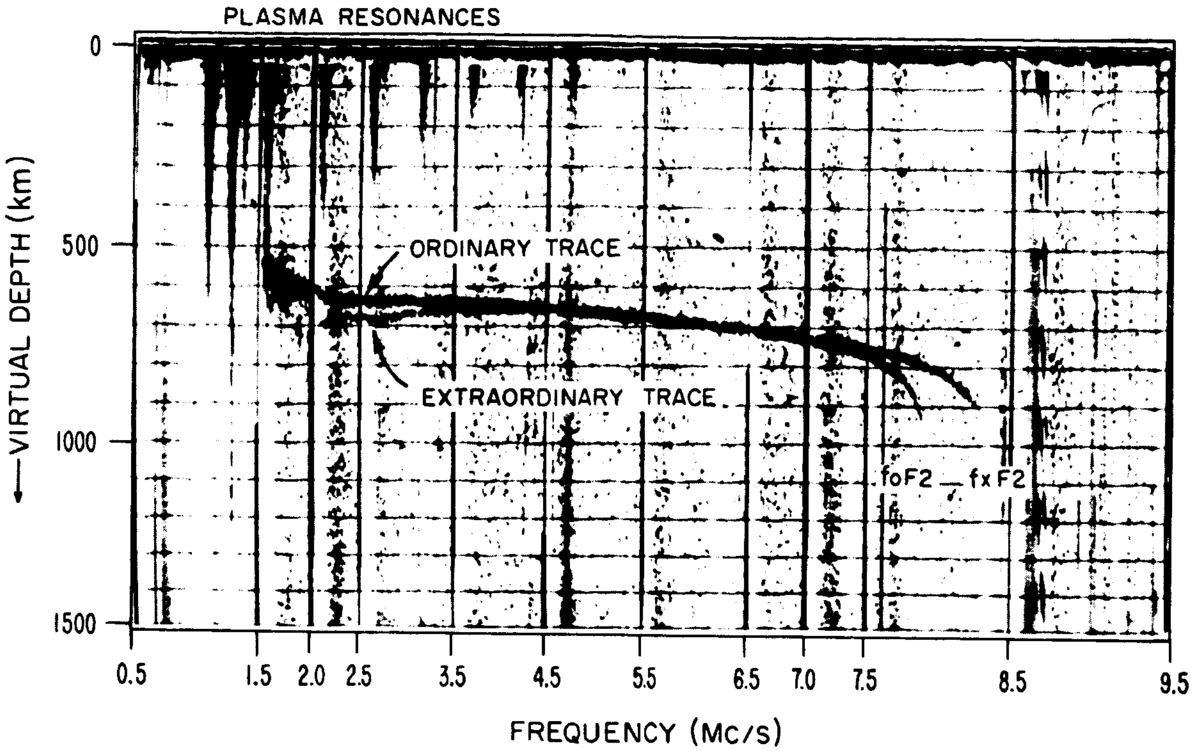
4b. Observations at f_H. See figure 4a.

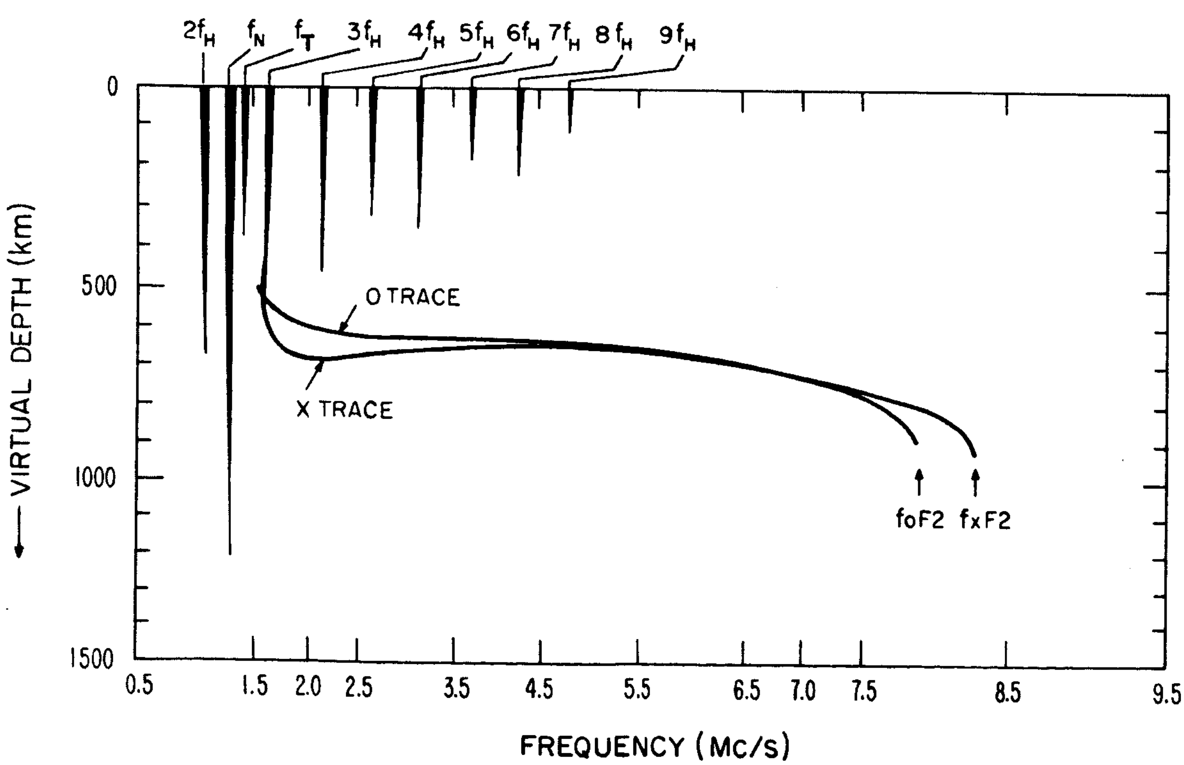
4c. Observations at f_T. See figure 4a.

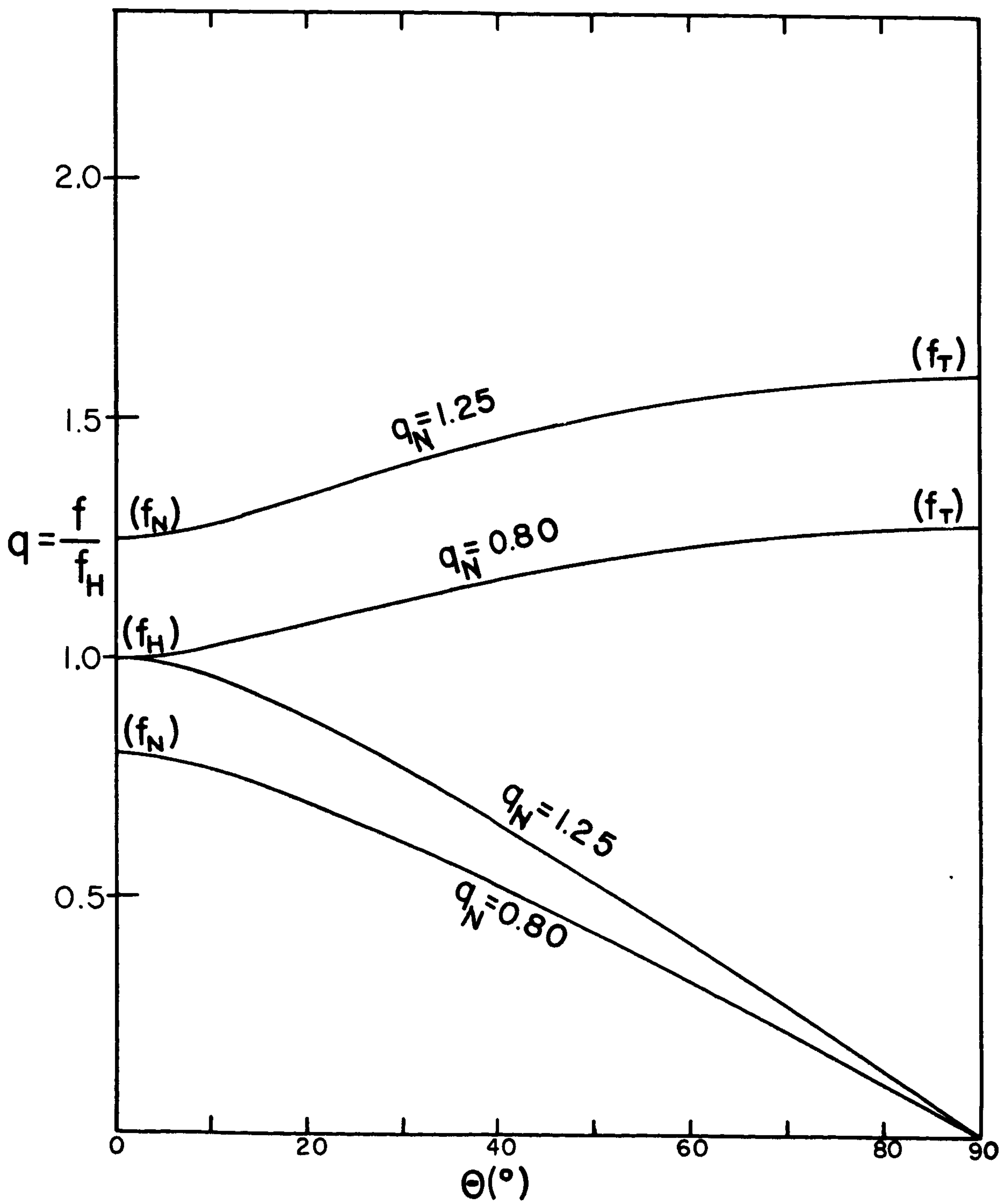
4d. Observations at $2f_H$. See figure 4a.

4e. Observations at $3f_H$. See figure 4a.

4f. Observations at $4f_H$. See figure 4a.


PLASMA RESONANCES
VIRTUAL DEPTH (km)
0
500
1000
1500
ORDINARY TRACE
EXTRAORDINARY TRACE
foF2
fxF2
0.5
1.5
2.0
2.5
3.5
4.5
5.5
6.5
7.0
7.5
8.5
9.5
FREQUENCY (Mc/s)

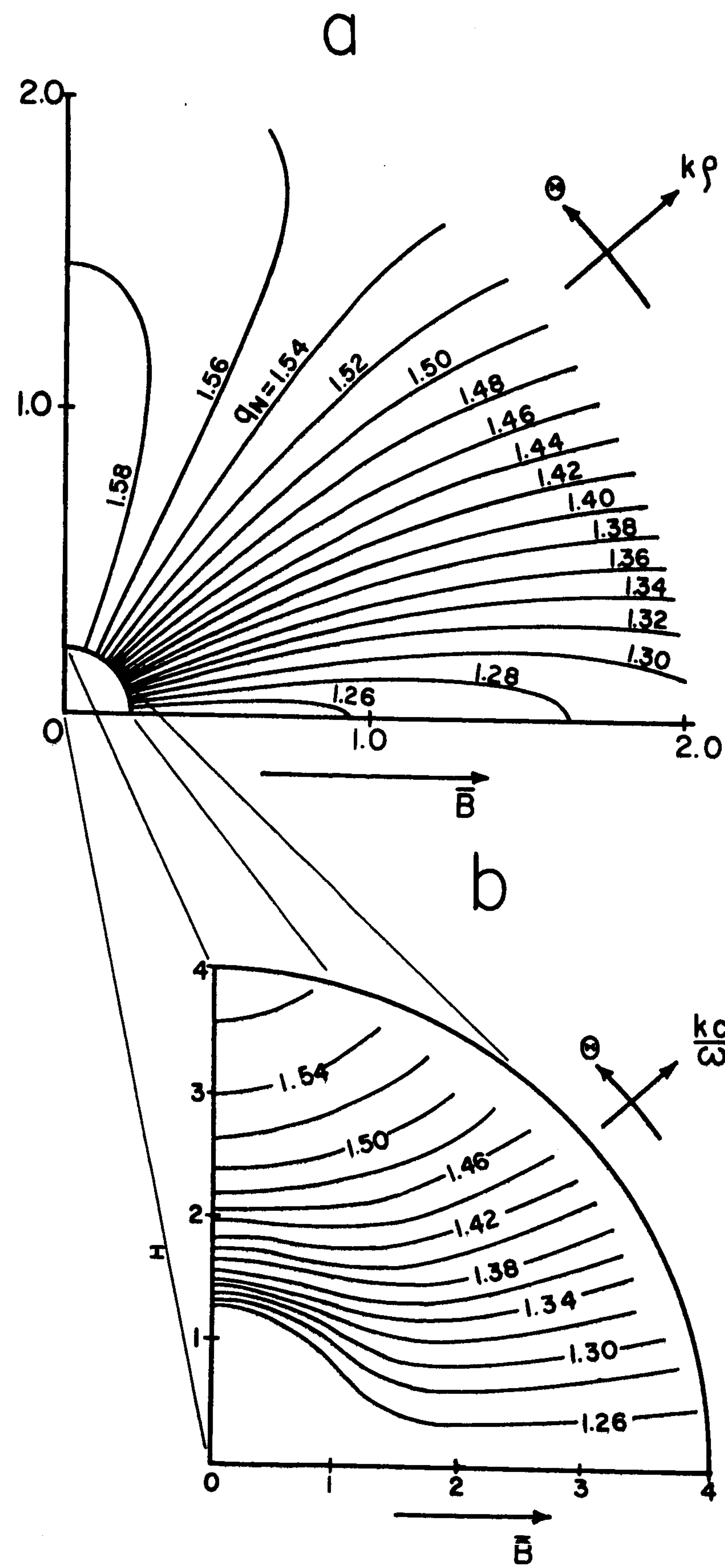

2fH
fN
fT
3fH
4fH
5fH
6fH
7fH
8fH
9fH
VIRTUAL DEPTH (km)
0
500
1000
1500
O TRACE
X TRACE
foF2
fxF2
0.5
1.5
2.0
2.5
3.5
4.5
5.5
6.5
7.0
7.5
8.5
9.5
FREQUENCY (Mc/s)



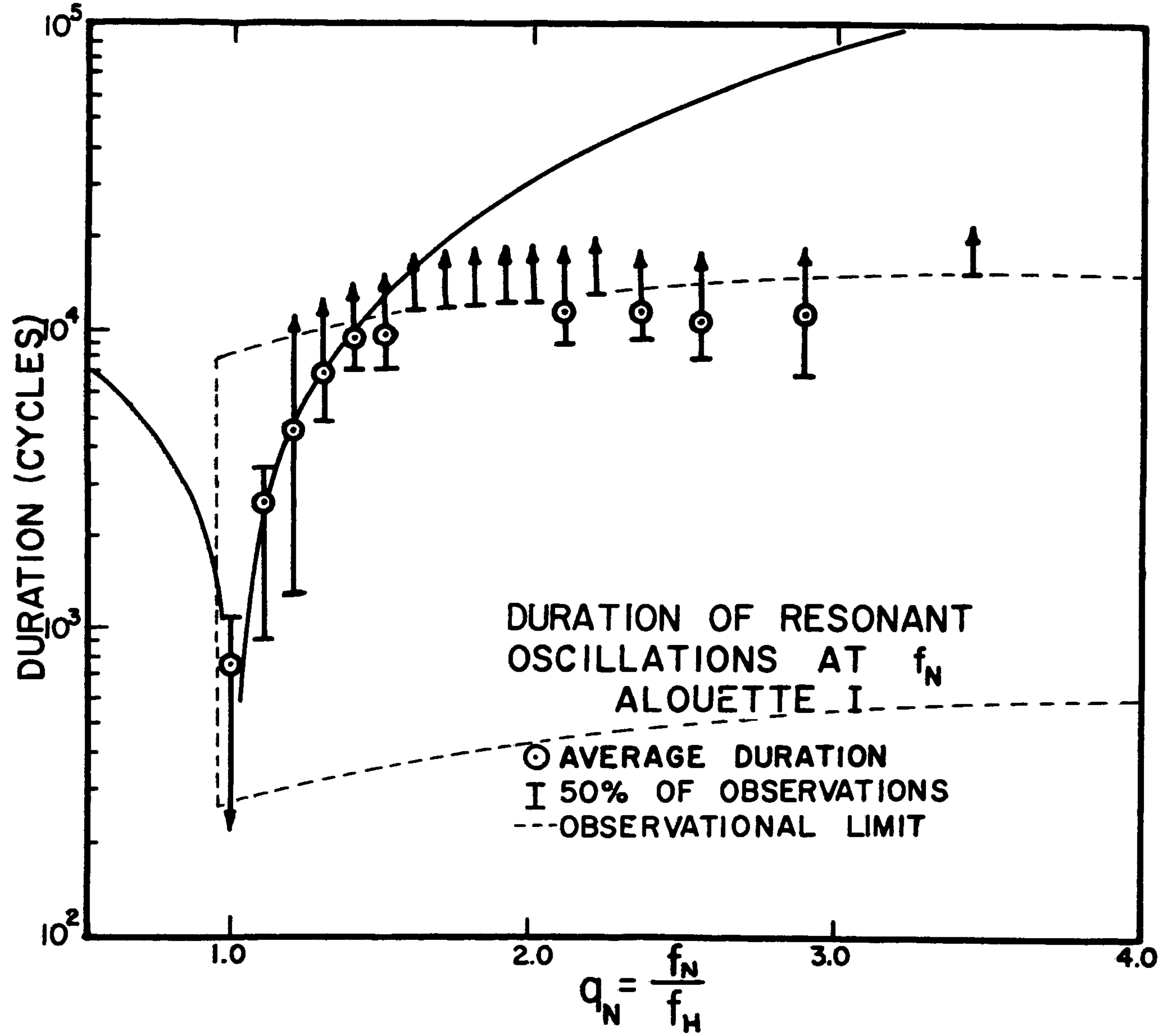
2.0
1.5
1.0
0.5
$q = \frac{f}{f_H}$
(f_N)
(f_H)
(f_N)
(f_T)
(f_T)
$q_N = 1.25$
$q_N = 0.80$
$q_N = 1.25$
$q_N = 0.80$
0
20
40
60
80
90
$\Theta(°)$

B- 46879

(3)

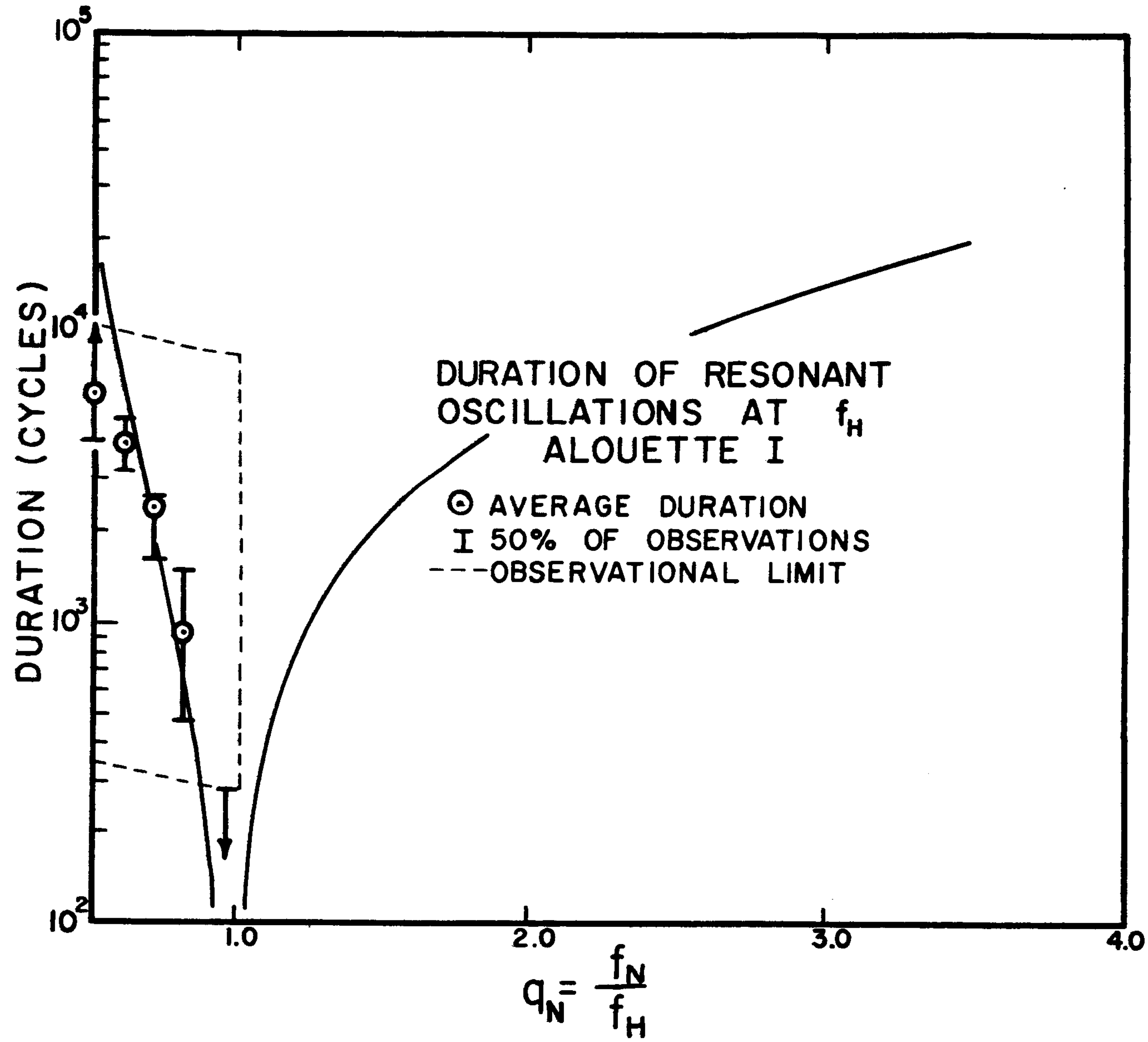

a
2.0
1.0
0
1.0
2.0
1.58
1.56
q_N=1.54
1.52
1.50
1.48
1.46
1.44
1.42
1.40
1.38
1.36
1.34
1.32
1.30
1.28
1.26
Θ
kρ
B
b
4
3
2
1
0
1.54
1.50
1.46
1.42
1.38
1.34
1.30
1.26
Θ
$\frac{kc}{\omega_N}$
B

B- 46878


DURATION OF RESONANT
OSCILLATIONS AT f_N
ALOUETTE I
AVERAGE DURATION
50% OF OBSERVATIONS
OBSERVATIONAL LIMIT
DURATION (CYCLES)
10^5
10^4
10^3
10^2
1.0
2.0
3.0
4.0
$q_N = \frac{f_N}{f_H}$

4a

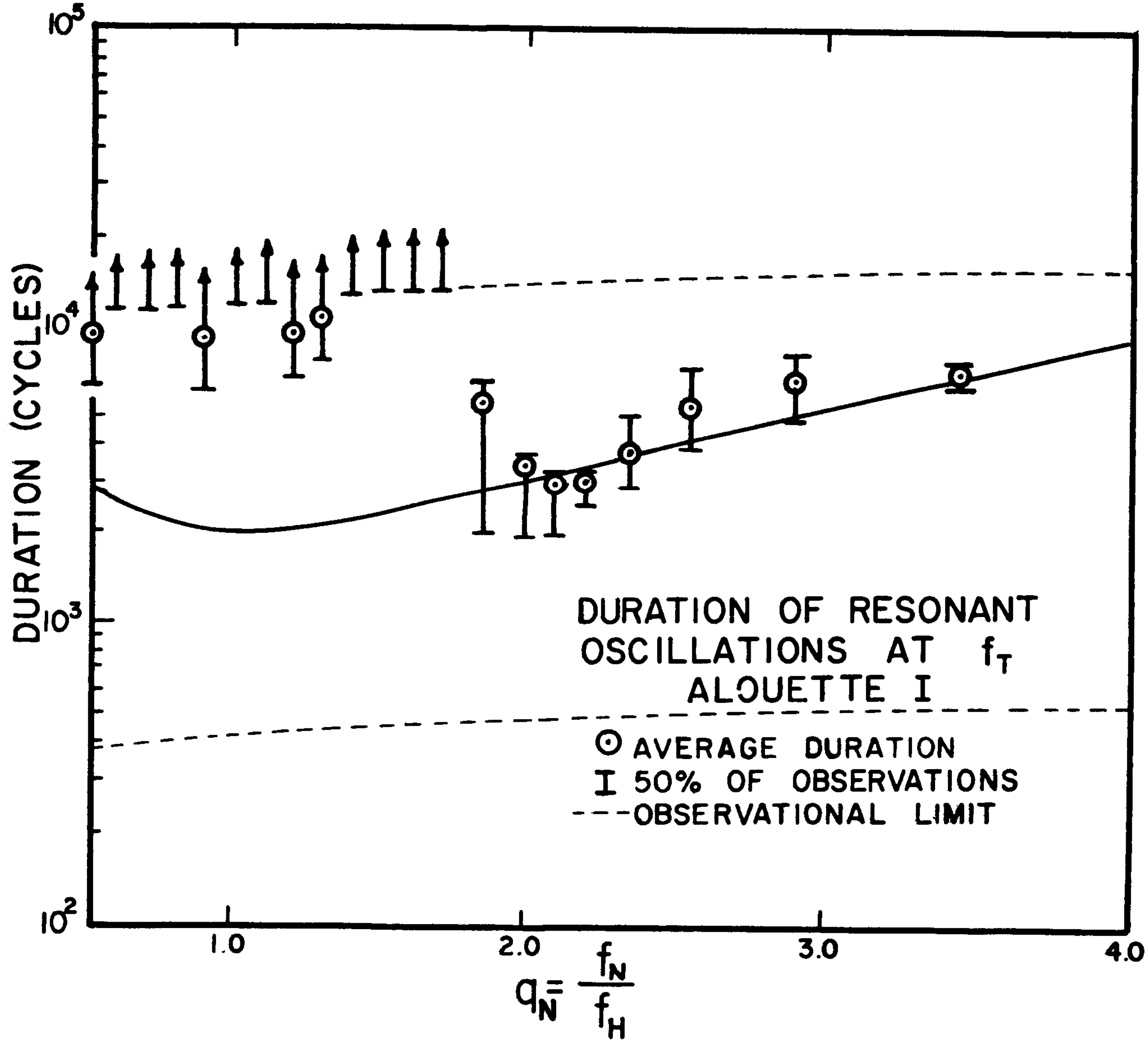
B- 46880


DURATION OF RESONANT
OSCILLATIONS AT f_H
ALOUETTE I
⊙ AVERAGE DURATION
I 50% OF OBSERVATIONS
---OBSERVATIONAL LIMIT
DURATION (CYCLES)
10^5
10^4
10^3
10^2
1.0
2.0
3.0
4.0
$q_N = \frac{f_N}{f_H}$

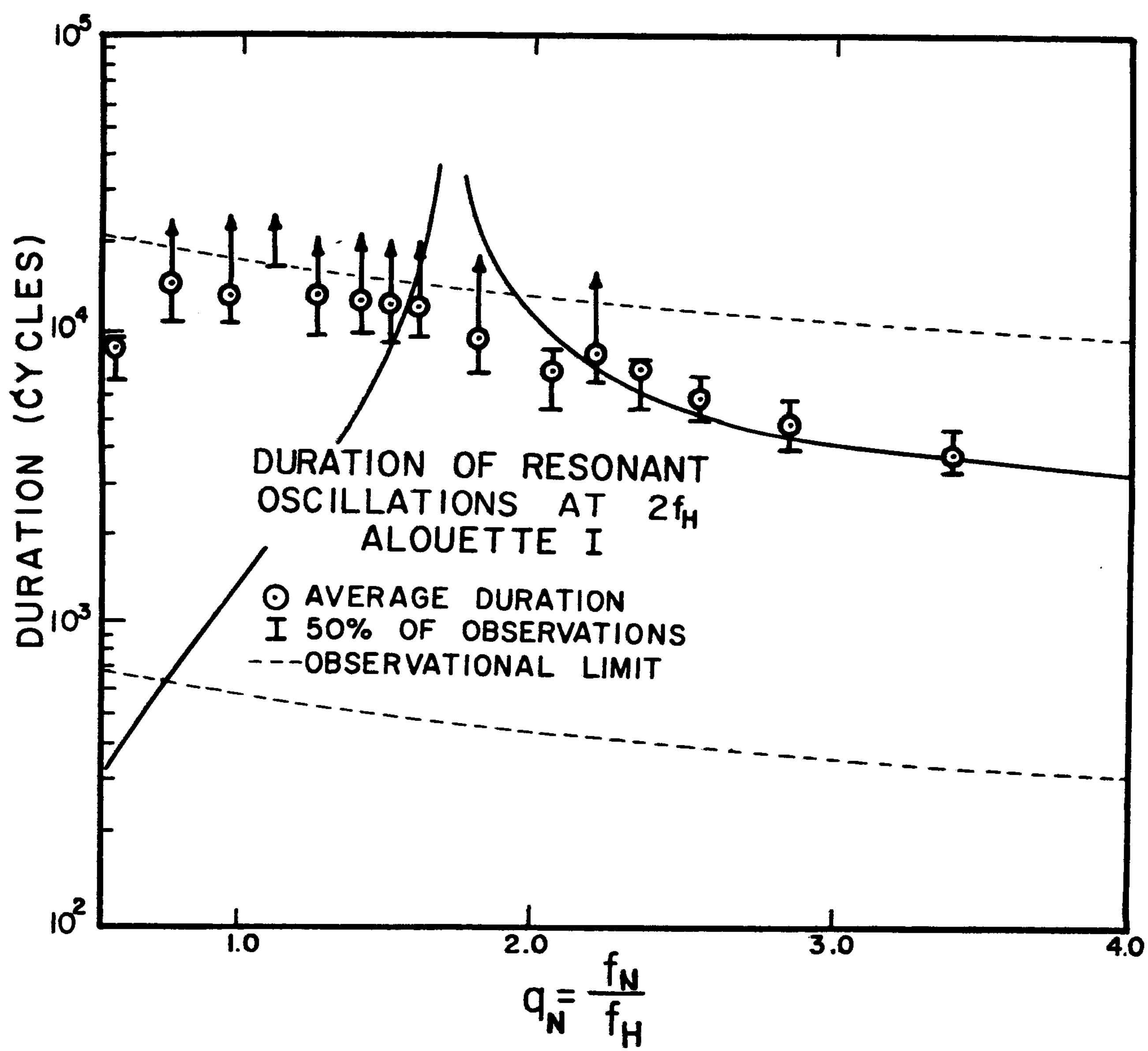
4b

B- 46881

4c

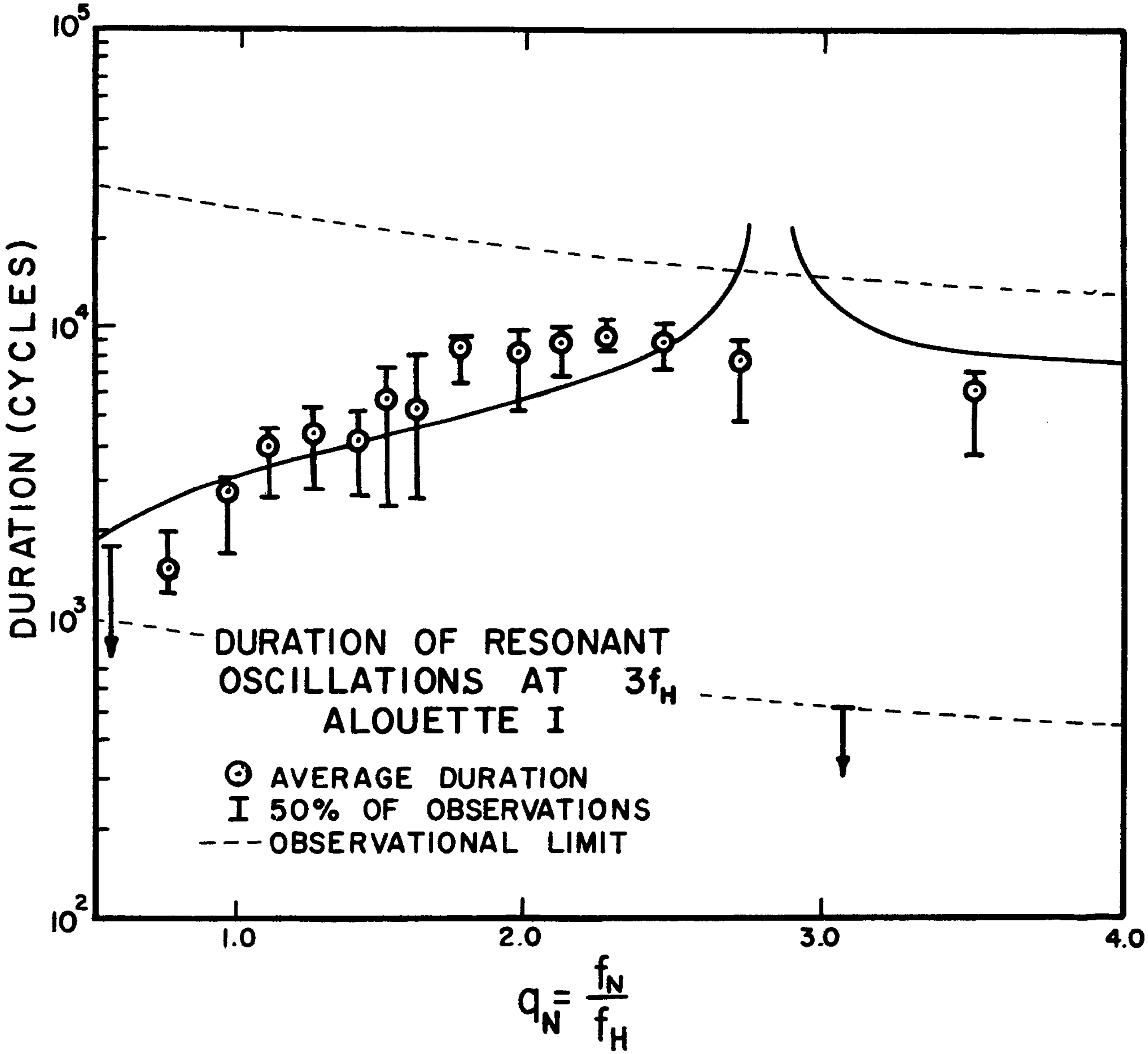

DURATION OF RESONANT
OSCILLATIONS AT f_T
ALOUETTE I
AVERAGE DURATION
50% OF OBSERVATIONS
OBSERVATIONAL LIMIT
DURATION (CYCLES)
10^5
10^4
10^3
10^2
1.0
2.0
3.0
4.0
$q_N = \frac{f_N}{f_H}$

B-46885


10^5
10^4
10^3
10^2
DURATION (CYCLES)
1.0
2.0
3.0
4.0
$q_N = \frac{f_N}{f_H}$
DURATION OF RESONANT
OSCILLATIONS AT $2f_H$
ALOUETTE I
AVERAGE DURATION
50% OF OBSERVATIONS
OBSERVATIONAL LIMIT

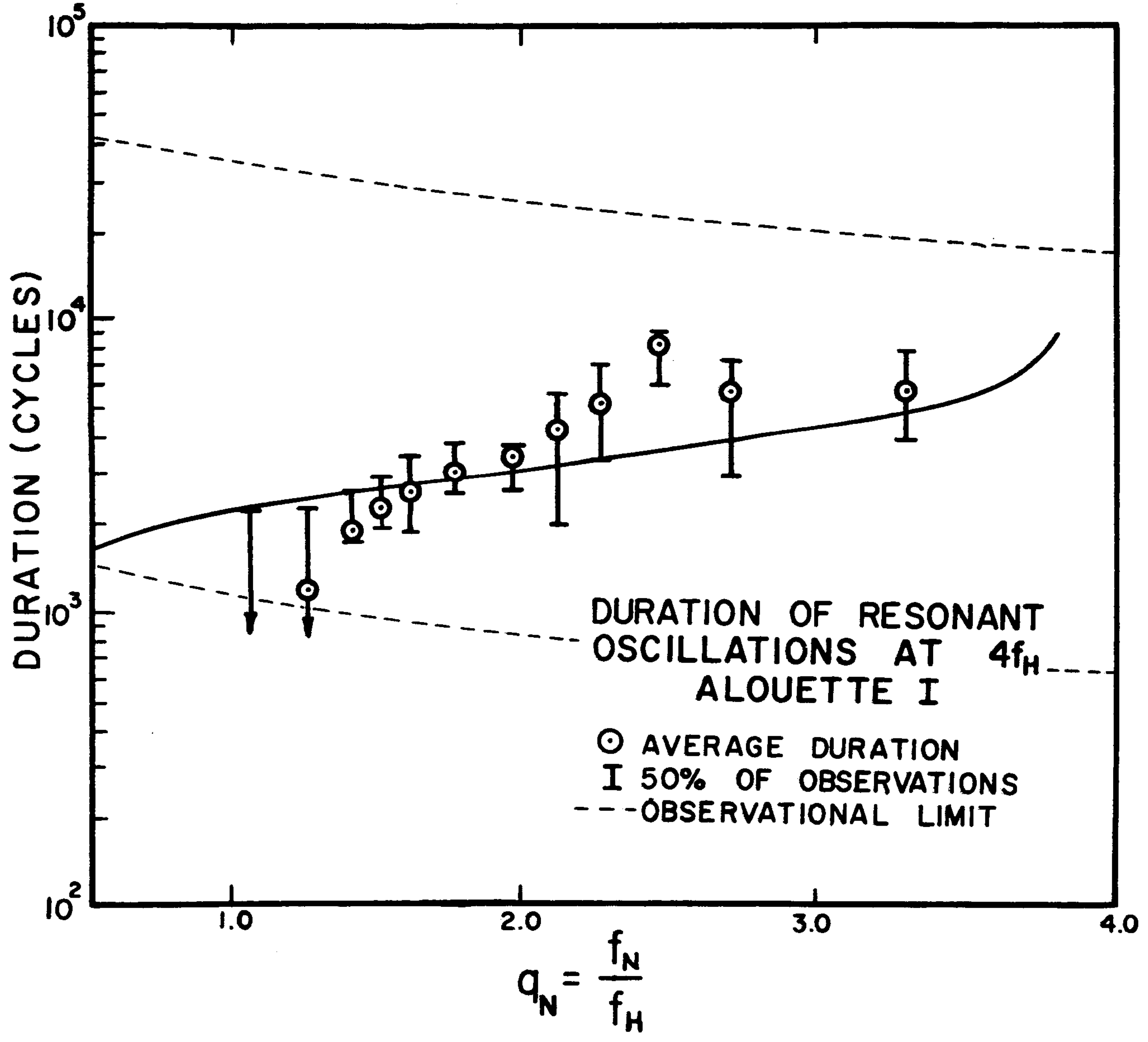
4d

B- 46882


DURATION OF RESONANT
OSCILLATIONS AT $3f_H$
ALOUETTE I
AVERAGE DURATION
50% OF OBSERVATIONS
OBSERVATIONAL LIMIT
DURATION (CYCLES)
$q_N = \frac{f_N}{f_H}$
10^5
10^4
10^3
10^2
1.0
2.0
3.0
4.0

4e

B- 46883


DURATION OF RESONANT
OSCILLATIONS AT $4f_H$
ALOUETTE I
AVERAGE DURATION
I 50% OF OBSERVATIONS
---OBSERVATIONAL LIMIT
DURATION (CYCLES)
10^5
10^4
10^3
10^2
1.0
2.0
3.0
4.0
$q_N = \frac{f_N}{f_H}$

4f

B-46884

	Relaxation Time (Cycles)	Angular Range (Radians)	Predicted Strength (θ_M = const) for Figure 4	Fractional Deviation of Frequency
	$\tau(hk, q_N)$	$\theta_m(\tau, q_N)$	$\tau(\theta_m, q_N)$	d_f
ω_N	$(3h^2k^2)^{-1}$	$\tau^{-1}\lvert q_N{}^2-1\rvert$	$\theta_m{}^{-1}\lvert q_N{}^2-1\rvert$	$\frac{1}{2}\tau^{-1}$
ω_H	$(10h^2k^2q_N{}^2)^{-1}$	$(5\tau/8)^{-1/4}\lvert q_N{}^2-1\rvert^{1/2}q_N{}^{-1}$	$\theta_m{}^{-4}\frac{8}{5}(q_N{}^2-1)^2q_N{}^{-4}$	$4(5\tau/2)^{-1/2}$
ω_T	$(3h^2k^2)^{-1}(q_N{}^2+1)\lvert q_N{}^2-3\rvert q_N{}^{-4}$	$\tau^{-1}(q_N{}^2+1)^2q_N{}^{-2}$	$\theta_m{}^{-1}(q_N{}^2+1)^2q_N{}^{-2}$	$\frac{1}{2}\tau^{-1}$
$n\omega_H$	$2^{n-1}n(n-2)!(k^2\rho^2)^{-(n-1)} \cdot \lvert q_N{}^{-2}-(n^2-1)^{-1}\rvert$	$2^{-9/2}\tau^{-\frac{2n-3}{2n-2}}\frac{n}{n-1} \cdot \left[n(n-2)!\lvert q_N{}^{-2}-(n^2-1)^{-1}\rvert\right]^{-\frac{1}{2n-2}}$	$\theta_m{}^{-\frac{2n-2}{2n-3}}\left(2^{-9/2}\frac{n}{n-1}\right)^{-\frac{2n-2}{2n-3}} \cdot \left[n(n-2)!\lvert q_N{}^{-2}-(n^2-1)^{-1}\rvert\right]^{-\frac{1}{2n-3}}$	$\frac{1}{2(n-1)}\tau^{-1}$

Table 1. Predicted relaxation times, angular ranges, and fractional deviations from nominal frequency for the plasma resonances observed by Alouette. For wavelengths as long as the sounder antenna, these quantities are the order of 10^4 cycles, 10^{-5} degrees, and 10^{-2} percent, respectively.